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RETROAZIMUTHALS ON A GLOBE AND THEIR IMAGES IN A MAP

Retroazimutály na glóbu a jejich obrazy v mapě. — V příspěvku se sleduje otázka, jak jsou po kulovém globu rozloženy hodnoty retroazimutu β (tj. azimutů ortodrom spojujících místa z kulového globu s cílem ležícím v zeměpisné šířce $\varphi = 50°$) a jak je můžeme vyhledat z tzv. retroazimutální mapy k tomu účelu sestrojené. Jsou připojeny pravoúhlé souřadnice x a y pro obrazy průsečíků geografické sítě této retroazimutální mapy v hustotě $\Delta\varphi = \Delta\lambda = 10°$. Této tabulky dá se užít ke konstrukci retroazimutálních map pro kterýkoli cíl podél 50°-rovnoběžky, tedy právě tak pro Prahu jako pro Krakov nebo Mohuč, Charkov nebo Lands End, Karagandu nebo Winnipeg atd. Hodí se také pro výpočet retroazimutů ve všech vrcholech 10° - sítě vůči cíli ležícímu na 50° - rovnoběžce. Poněvadž cotg β = x : y, dostáváme je jednodušeji nežli řešením sférického trojúhelníka OPM.

Points on a spherical globe, in a given direction (or azimuth) α from a given point $O(\varphi_0, \lambda_0)$ constitute an orthodrome. The arc of an orthodrome with end points O and a given $M(\varphi, \lambda)$, intersect the meridians λ_0 to λ under different angles. Thus the azimuth varies from point to point on the orthodrome; the corresponding convex angle, measured from north on the incident meridian, is then $180° - \beta$, where β is the so-called retrograde azimuth, or *retroazimuth,* of the arc MO of an orthodrome.

We shall be interested in the question as to the distribution of retroazimuth values β on a spherical globe, and their determination from a conveniently constructed *retroazimuthal map*.

Curves of constant retroazimuth β on a spherical globe will be called **retro-***azimuthals;* from each point $M(\varphi, \lambda)$ on a retroazimuthal, orthodromic arcs in the direction β are directed towards the given point $O(\varphi_0, \lambda_0)$. The angles β are measured from the north direction of the meridian at M to the orthodromic arcs leading from M to O in such sense that β is not greater than 180°. Thus all retroazimuth values are in the interval $<0°, 180°>$. With the given point O there is associated the point $O'(-\varphi_0, -\lambda_0)$ opposite to O on the globe. Let the orthodromic arcs from M have azimuths β' with respect to O'; if these are measured from meridional south so as not to exceed 180°, then $\beta = \beta'$ for every point M of the globe. The meridional circle passing through both O, O' divides the globe into two hemispheres, an eastern and a western. Points on the globe, symmetric about the plane of this meridional circle, have complementary azimuths. Thus it suffices to consider the distribution of retroazimuths only on one hemisphere, i.e. to restrict the values of the coordinate λ to the interval $<0°, 180°>$. Furthermore, the distribution of retroazimuths on the globe is also centrally symmetric; and thus, on each hemisphere, this distribution is spherically centrally symmetric with respect to the spherical center R(0°, 90°) of the hemisphere. Therefore it is sufficient to consider only a certain half part of the hemisphere (cf. fig. 1).

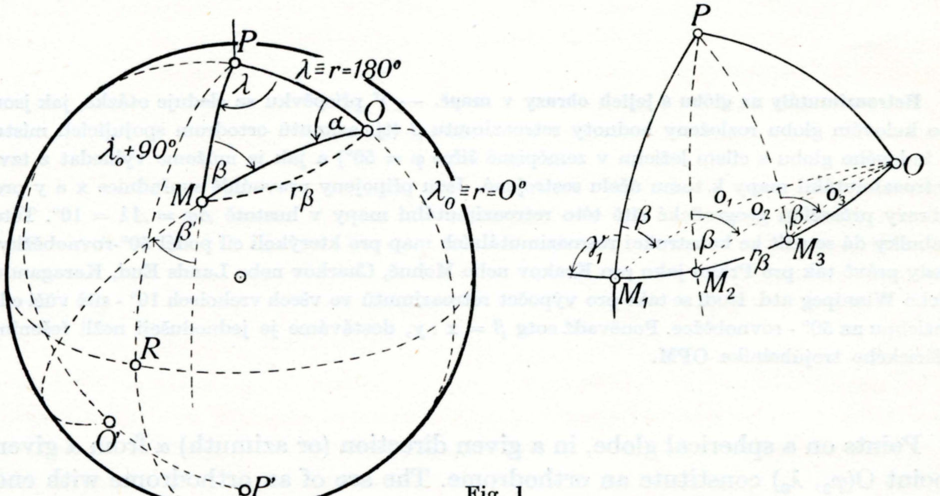

Fig. 1

If P and P′ are poles on the globe, then from the spherical triangle MOP (or MOP′) there follows

$$\cot \beta = \frac{1}{\sin \lambda} (\tan \varphi_0 \cos \varphi - \sin \varphi \cos \lambda) . \tag{1}$$

This is the defining equation of retroazimuthals; φ, λ are independent variables, the coordinates of points on the curve, and β is a parameter whose values, in $<0°, 180°>$, distinguish between distinct retroazimuthals. A meridian λ intersects a retroazimuthal under an azimuth γ; it is measured from the same direction on the meridian as β or β', but in the opposite sense. Also

$$\tan \gamma = \frac{\tan \varphi_0 \sin \varphi \cos \varphi - \cos^2 \varphi \cos \lambda}{\cot \lambda (\sin \varphi \cos \lambda - \tan \varphi_0 \cos \varphi) - \sin \varphi} ; \tag{2}$$

here γ also determines the tangents to the retroazimuthal. Equations (1) and (2) determine retroazimuthals analytically.

The meridional arcs OP and O′P′ correspond to the retroazimuthal $\beta = 180°$, the arcs OP′ and O′P to the retroazimuthal $\beta = 0°$. At the pole P we have $\beta = 180° - \lambda$, at P′ $\beta = \lambda$, so that at either pole, β takes on all values from 0° to 180°; thus both poles belong to all retroazimuthals. At the points O and O′, $\cot \beta = 0/0$ is indeterminate, and these two points also belong to all retroazimuthals. From the values tan γ assumes at these points it follows that the retroazimuthals approach these points under different azimuths β; thus the points P, P′, O, O′ are all nodal points of the system of retroazimuthals.

Each retroazimuthal decomposes into two components. For the retroazimuthal $\beta = 90°$ (considered on the whole globe), one component contains the points P and O, and at these points it is orthogonal to the meridian $\lambda = 0°$; the second component contains P′ and O′ and is orthogonal to the meridian $\lambda = 180°$. All other retroazimuthals have angular points at the poles and at O and O′. The retroazimuthal $\beta = 90° - \varphi_0$ is specific in that one component passes through P, P′, the second through O, O′; they intersect at the spherical centers R of both hemispheres, which are thus double points of the complete retroazimuthal. The first component is tangent, at P, to the meridian $\lambda = 90° + \varphi_0$, and at P′ to the meridian $\lambda = 90° - \varphi_0$. Thus, on either hemisphere, this component lies within the spherical bi-angle bounded by the mentioned meridians. At the points R this component has azimuth $\gamma = \frac{1}{2}\varphi_0$.

This component (i.e. the component of the retroazimuthal $\beta = 90° - \varphi_0$ passing through the poles), divides the globe into two parts; in one, retroazimuthals are taken with respect to O, in the other, with respect to O′. The second component (passing through O, O′) has, on both hemispheres, the azimuth $\gamma = 90° - \frac{1}{2}\varphi_0$ at the points R, and azimuth $\gamma = 90° + \frac{1}{2}\varphi_0$ at the points O, O′. This component lies within the strip bounded by the parallels φ_0 and $-\varphi_0$. The two components divide the globe into four bi-angular sectors. An opposite pair of these contains arcs of the retroazimuthal $\beta = 90°$, and consists of all points M with β within $\langle 90° - \varphi_0, 180° \rangle$. The second pair of opposite sectors consists of all points M with β within $\langle 90° - \varphi_0, 0° \rangle$.

Now project the points of each retroazimuthal (on the eastern or western hemisphere) along normals to the globe, to a height proportional to the corresponding values of β. There results the surface of interpolation of the field of retroazimuthals; it is essentially of hyperbolic type, since above the spherical center R there appears a simple saddle point, with relative maxima over the arcs OP and OP′, and relative minima over O′P and OP′. The asymptotic curves are perpendicular at the saddle point. These phenomena are preserved by a stereographic — and hence conformal — mapping of the retroazimuthal field into the plane tangent to the globe at R.

The spherical globe with retroazimuthal field may of course be mapped onto any kind of map; however, only for the so-called retroazimuthal maps it is possible to determine easily the retroazimuths β from a point $M(\varphi, \lambda)$ towards a point $O(\varphi_0, \lambda_0)$ given in advance. In current cartographic literature, the description of retroazimuthal maps is rather summary. The purpose of the present paper is their detailed study, including mention of methods and instruments for finding retroazimuth values of any point on the globe using retroazimuthal maps.

In practice, retroazimuthal maps are used not for the complete globe, but only for the bi-angular sector bounded by the meridians $\lambda_0 - 90°$, $\lambda_0 + 90°$, and with central meridian λ_0.

Retroazimuths β, both positive and negative, may be determined from (1), as angles between 0° and 180°, measured north of meridians through M towards O.

The image of the geographical network in a retroazimuthal map is unexpectedly complex. This is caused by the fact that the directions of retroazimuthals at O and the directions of their images in the map are different. The images of meridians are parallel straight lines, their distances from the central meridian $\lambda_0 = 0°$ are on the Y axis; the X and Y axes intersect at the image of the point O. In this coordinate system, the map of the spherical bi-angular sector had determining equations

$$\begin{aligned} x &= r \cos \varphi_0 \operatorname{arc} \lambda \ , \\ y &= x \operatorname{cotan} \beta \ , \end{aligned} \tag{3}$$

where the independent variables φ, λ are coordinates of any point M of the spherical sector on the globe.

From the second of these equations it follows that points M with fixed β (i.e. on a given retroazimuthal on the globe) and with λ of fixed sign, have map images on a ray with vertex in the origin. Such rays are then images of retroazimuthals; and thus we may assign to each of them the corresponding value of β, and conversely, use them to determine retroazimuths of points on the globe.

Thus, in the map constructed for $\varphi_0 = 50°$, images of retroazimuthals are rays incident with the given point O. With the assumption $\lambda_0 = 15°$ E. Gr. (i.e. the given point has approximately the location of Prague), contours of continents and islands were drawn in the network corresponding to (3). If, in this map, a straight line segment is drawn from the image of any point to the image of Prague, then the reading on an angular scale of this segment is the retroazimuth of Prague at the given point on the globe. The map images of retroazimuthals have inclinations to the image of the central meridian in agreement with the characteristics of the retroazimuthal, measured clockwise from meridional south on an angular scale.

The images of geographical parallels φ may again be determined from (3) with φ kept constant. For $|\varphi_0| \neq 0$ they are all curvilinear and symmetric with respect to the Y axis. This also holds for the parallel φ_0; the images of poles are similar singular curves. The images of parallels φ and $-\varphi$ intersect on the images of meridians 90° and $-90°$. In considering their curvature, let $\bar{\varphi}$ be the parallel with $\tan \bar{\varphi} = (1 - \frac{\pi}{2}) \operatorname{cotan} \varphi_0$. Then for $\varphi_0 > 0°$ all parallels $\varphi > \bar{\varphi}$ are convex in the south direction, all parallels $\varphi < \bar{\varphi}$ are convex in the north direction; for $\varphi_0 < 0°$ the seare interchanged. The parallel $\bar{\varphi}$ thus divides the biangular sector into two parts whose images are not disjoint.

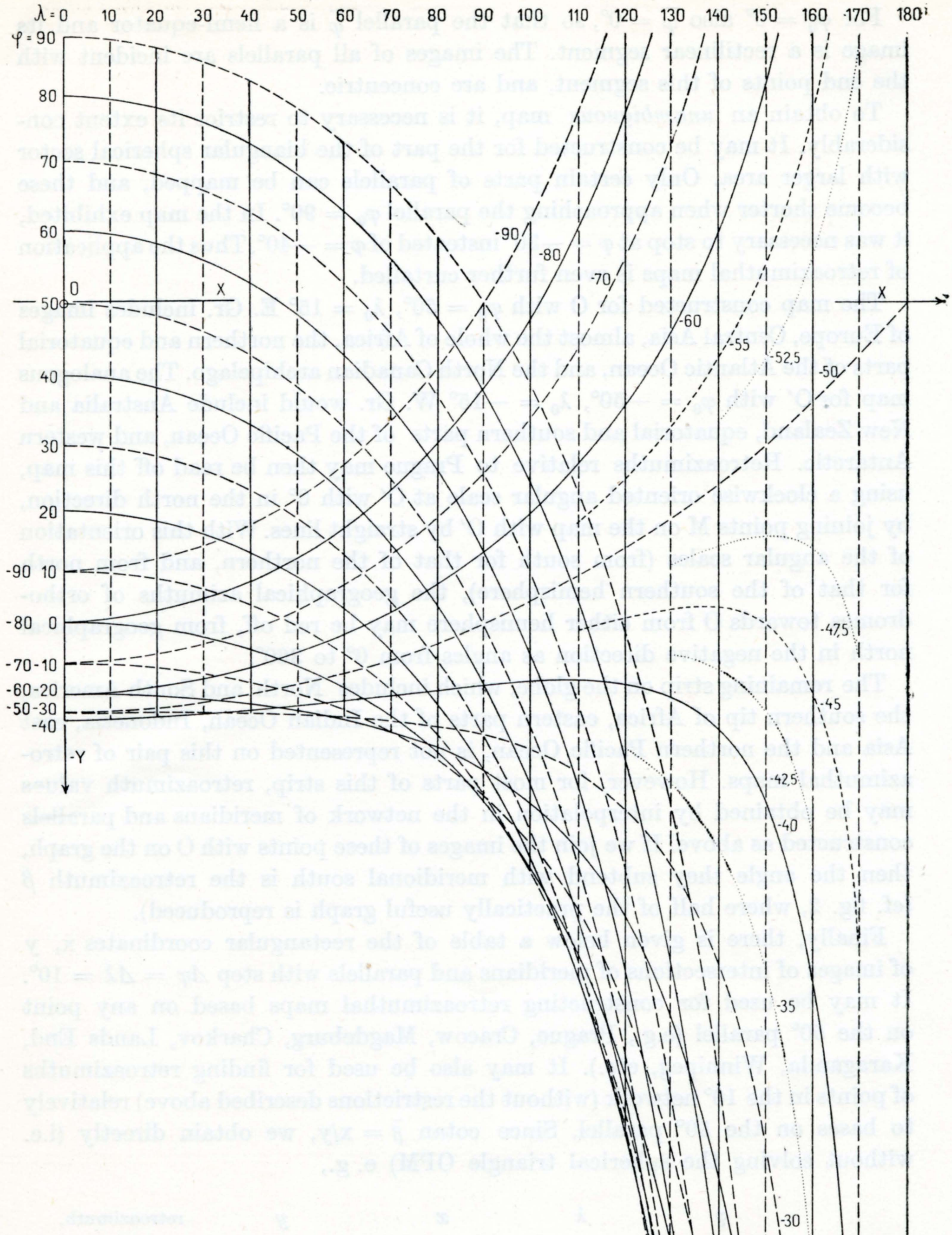
λ = 0
10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
160
170
180
φ = 90
80
70
60
50
40
30
20
-90 10
-80 0
-70 -10
-60 -20
-50 -30
-40
O
X
X
Y
-90
-80
-70
-60
-55
-52,5
-50
-47,5
-45
-42,5
-40
-35
-30

Fig. 2

For $\varphi_0 = 0°$ also $\bar{\varphi} = 0°$, so that the parallel $\bar{\varphi}$ is a hemi-equator and its image is a rectilinear segment. The images of all parallels are incident with the end points of this segment, and are concentric.

To obtain an *unambiguous* map, it is necessary to restrict its extent considerably. It may be constructed for the part of the biangular spherical sector with larger area. Only certain parts of parallels can be mapped, and these become shorter when approaching the parallel $\varphi_0 = 90°$. In the map exhibited, it was necessary to stop at $\varphi = -30°$ insteated of $\varphi = -40°$. Thus the application of retroazimuthal maps is even further curtailed.

The map constructed for O with $\varphi_0 = 50°$, $\lambda_0 = 15°$ E. Gr. includes images of Europe, Central Asia, almost the whole of Africa, the northern and equatorial parts of the Atlantic Ocean, and the North Canadian archipelago. The analogous map for O′ with $\varphi_0 = -50°$, $\lambda_0 = -15°$ W. Gr. would include Australia and New Zealand, equatorial and southern parts of the Pacific Ocean, and western Antarctic. Retroazimuths relative to Prague may then be read off this map, using a clockwise oriented angular scale at O′ with 0° in the north direction, by joining points M on the map with O′ by straight lines. With this orientation of the angular scales (from south for that of the northern, and from north for that of the southern hemisphere), the geographical azimuths of orthodromes towards O from either hemisphere may be red off, from geographical north in the negative direction as angles from 0° to 360°.

The remaining strip on the globe, which includes North and South America, the southern tip of Africa, eastern parts of the Indian Ocean, Indonesia, east Asia and the northern Pacific Ocean, is not represented on this pair of retroazimuthal maps. However, for most parts of this strip, retroazimuth values may be obtained by interpolation in the network of meridians and parallels constructed as above. If we join the images of these points with O on the graph, then the angle they subtend with meridional south is the retroazimuth β (cf. fig. 2, where half of the practically useful graph is reproduced).

Finally, there is given below a table of the rectangular coordinates x, y of images of intersections of meridians and parallels with step $\Delta\varphi = \Delta\lambda = 10°$. It may be used for constructing retroazimuthal maps based on any point on the 50° parallel (e.g., Prague, Cracow, Magdeburg, Charkov, Lands End, Karaganda, Winnipeg, etc.). It may also be used for finding retroazimuths of points in the 10° network (without the restrictions described above) relatively to bases on the 50° parallel. Since cotan $\bar{\beta} = x/y$, we obtain directly (i.e. without solving the spherical triangle OPM) e. g.,

	φ	λ	x	y	retroazimuth.
M_1	70°	60°	428,86	30,82	265°53′20″
M_2	—60°	160°	1143,65	726,87	237°34′0″
M_3	—30°	—90°	—643,30	—663,94	44°5′40″
M_4	80°	—170°	—1215,17	—8234,62	8°23′40″

	$\lambda = 0°$	10°	20°	30°	40°	50°
φ	$x =$ 0,00	71,48	142,95	214,43	285,91	357,39
90°	$y =$ 409,53	405,36	392,76	371,41	340,73	299,89
80°	318,56	314,02	300,30	277,01	243,51	198,78
70°	217,91	213,14	198,71	174,23	138,88	91,64
60°	110,64	105,82	91,09	66,10	30,04	— 18,29
50°	0,00	— 4,79	— 19,31	— 44,02	— 79,72	—127,66
40°	—110,63	—115,21	—129,11	—152,78	—187,05	—233,15
30°	—217,91	—222,14	—234,00	—256,92	—288,70	—331,56
20°	—318,56	—322,32	—333,75	—353,25	—381,58	—419,90
10°	—409,53	—412,70	—422,35	—438,84	—462,87	—495,48
0°	—488,06	—490,54	—498,12	—511,10	—530,08	—555,99
—10°	—551,77	—553,48	—558,76	—567,83	—581,20	—599,67
—20°	—598,70	—599,60	—602,41	—607,31	—614,65	—625,00
—30°	—627,44	—627,51	—627,77	—628,33	—629,43	—631,50
—40°	—637,12	—636,34	—634,04	—630.26	—625,08	—618,68
—50°	—627,44	—625,84	—621,06	—613,04	—601,74	—587,11
—60°	—598,70	—596,33	—589,21	—577,20	—560,12	—537,71
—70°	—551,76	—548,69	—539,44	—523,81	—501,48	—471,96
—80°	—488,07	—484,39	—473,30	—454,52	—427,60	—391,88
—90°	—409,53	—405,36	—392,76	—371,41	—340,73	—299,89

	$\lambda = 60°$	70°	80°	90°	100°	110°
φ	$x =$ 428,86	500,34	571,82	643,30	714,77	786,25
90°	$y =$ 247,60	182,11	100,83	0,00	—125,04	— 286,17
80°	141,36	69,15	— 20,87	—133,13	—274,32	— 454,98
70°	30,82	— 45,80	—141,92	—262,21	—414,27	— 609,96
60°	— 80,65	—159,56	—258,67	—383,32	—541,64	— 746,41
50°	—189,68	—268,38	—367,56	—492,79	—652,54	— 860,18
40°	—292,93	—368,03	—465,27	—587,28	—743,61	— 947,81
30°	—387,30	—458,48	—548,85	—663,94	—812,11	—1006,65
20°	—469,89	—533,99	—615,76	—710,39	—855,91	—1034,90
10°	—538,20	—593,29	—663,96	—754,00	—873,71	—1031,70
0°	—590,16	—634,55	—691,97	—766,65	—864,97	— 997,15
—10°	—624,19	—656,54	—698,97	—755,00	—829,95	— 932.31
—20°	—639,26	—658,57	—684,73	—720,41	—769,70	— 839,14
—30°	—634,90	—640,56	—649,65	—663,94	—686,07	— 720,48
—40°	—611,25	—603,15	—522,03	—492,79	—459,44	— 421,74
—50°	—569,02	—547,38	—594,89	—587,28	—581,59	— 579,91
—60°	—509,52	—474,99	—433,31	—383,33	—323,34	— 250,75
—70°	—434,52	—388,15	—331,42	—262,21	—177,40	— 72,13
—80°	—346,33	—289,53	—219,46	—133,13	— 26,09	— 108,66
—90°	—247,60	—182,11	—100,83	0,00	126,04	286,17

	λ = 120°	130°	140°	150°	160°	170
φ	x = 857,73	929,21	1000,68	1072,16	1143,64	1215,12
90°	y = — 495,21	— 779,70	—1192,54	—1857,05	—3142,10	— 6891,18
80°	— 692,65	—1018,90	—1496,59	—2272,62	—3786,38	— 8234,62
70°	— 868,03	—1227,10	—1755,16	—2619,07	—4315,53	— 9327,76
60°	—1019,03	—1398,06	—1960,42	—2886,02	—4713,64	—10137,65
50°	—1138,05	—1526,49	—2106,07	—3065,20	—4968,44	—10639,23
40°	—1222,49	—1608,56	—2187,75	—3151,30	—5072,29	—10817,67
30°	—1269,80	—1641,77	—2202,99	—3141,67	—5022,14	—10667,64
20°	—1278,51	—1625,09	—2151,26	—3036,53	—4819,27	—10193,21
10°	—1248,40	—1559,03	—2034,18	—2839,17	—4470,05	— 9409,22
0°	—1180,32	—1445,59	—1855,27	—2555,50	—3984,93	— 8339,23
—10°	—1078,41	—1288,24	—1620,01	—2194,20	—3378,78	— 7015,94
—20°	— 939,77	—1091,74	—1335,52	—1766,24	—2669,96	— 5479,44
—30°	— 774,59	— 862,08	—1010,45	—1284,60	—1880,00	— 3776,39
—40°	— 585,86	— 606,19	— 654,65	— 763,89	—1032,89	— 1958,59
—50°	— 379,35	— 331,93	— 279,02	— 220,07	— 154,48	— 81,45
—60°	— 161,30	— 47,55	105,13	330,50	726,87	1798,35
—70°	— 61,65	238,26	486,08	871,02	1589,69	3623,43
—80°	282,72	516,84	852,25	1385,08	2402,40	5338,37
—90°	495,21	779,70	1192,54	1857,05	3142,10	6891,18